Linear Algebra II

25/06/2021, Friday, 08:30 – 11:30 (deadline for handing in: 11:50)

- This Take-Home Exam is 'open-book', which means that the book as well as lecture notes may be used as a reference. The exam contains 5 problems.
- For handing in the exam, the use of electronic devices is of course allowed. The student is fully responsible for handing in his/her complete work before the deadline. You are asked to upload your answers as a **pdf-file**.
- Every student must upload the signed declaration before the start of the exam. An exam will not be graded in case the signed declaration has not been uploaded. After grading, short discussions with (a selection of) students will be held to check for possible fraud.
- Write your name and student number on each page!

1 $(6+6+6 = 18$ pts) **Least squares approximation**

Consider the vector space $\mathbb{R}^{2\times 2}$. Let S be the linear subspace of $\mathbb{R}^{2\times 2}$ spanned by the matrices M_1 and M_2 given by

$$M_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

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(a) Show that $\langle A, B\rangle := \operatorname{trace}(A^T B)$ defines an inner product on $\mathbb{R}^{2\times 2}$.

(b) Determine an orthonormal basis for the subspace S with respect to this inner product

(c) Determine the best approximation of the matrix $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the subspace S. Here, a, b, c and d are given real numbers.

2 $(4+3+4+3+4 = 18 \text{ pts})$ **Eigenvalues**

Assume $n > 1$ an integer. Let $x \in \mathbb{R}^n$ with $x \neq 0$. Define the $n \times n$ matrix M by $M := xx^\top$.

(a) Find a basis of $R(M)$. Explain your answer.

(b) Determine the rank of M. Explain your answer.

(c) Show that $x_1^2 + x_2^2 + \ldots + x_n^2$ is an eigenvalue of M. Determine a corresponding eigenvector.

(d) Determine all eigenvalues of M. Motivate your answer carefully.

(e) Determine the Jordan Form of M. Explain your answer.

3 $(4+5+5+4 = 18 \text{ pts})$ **Positive definite matrices**

Let A be a real $n \times n$ matrix. Consider the linear matrix inequality

$$X - A^\top X A > 0$$

in the unknown symmetric matrix $X \in \mathbb{R}^{n\times n}$.

(a) Show that if the inequality has a positive definite solution solution X, then every eigenvalue λ of A satisfies $|\lambda| < 1$.

(b) Assume all eigenvalues λ of A satisfy $|\lambda| < 1$. This implies that $\lim_{k\to\infty}(A^\top)^k A^k = 0$, and the infinite sum

$$X := \sum_{k=0}^{\infty} (A^\top)^k A^k \quad (\star)$$

converges. Now, consider the linear equation

$$X - A^\top X A = I$$

Show that the infinite sum $(\star)$ is a solution of this linear equation.

(c) Prove that X given by $(\star)$ is positive definite.

(d) Prove that the linear matrix inequality $X - A^\top X A > 0$ has a positive definite solution X if and only if all eigenvalues λ of A satisfy $|\lambda| < 1$.

4 $(6+6+6=18 \text{ pts})$ **Cayley-Hamilton**

Consider the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{bmatrix},$$

where a, b and c are real numbers.

(a) Determine the characteristic polynomial of A.

(b) Do there exist a, b and c such that $2A^2 = I + A^3$? Explain.

(c) Compute a, b and c such that $A^{12} = A^8$.

5 $(2+3+4+4+5=18 \text{ pts})$ **Jordan Form**

Let $A \in \mathbb{C}^{3\times 3}$ be defined as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix}.$$

Here $a, b \in \mathbb{C}$ are given constants.

(a) Determine for all values of a and b the eigenvalues of A and their corresponding algebraic multiplicities.

(b) Give necessary and sufficient conditions on a and b so that A is in Jordan normal form.

(c) Assume $a \neq 0$. Determine the geometric multiplicity of each of the eigenvalues.

(d) Assume that $a = 0$ and $b \neq 0$. Determine the the geometric multiplicity of each of the eigenvalues

(e) Determine for all values of a and b the Jordan Form of A..

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10 pts free